Scalar–Flat Lorentzian Einstein–Weyl Spaces

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Abstract

We find all three-dimensional Einstein–Weyl spaces with the vanishing scalar curvature.

Three-dimensional Lorentzian Einstein metrics have constant curvature. In order to construct non-trivial gravitational models in three dimensions, one should therefore look at conformal geometries involving a non-metric connection. Einstein–Weyl geometries appear quite naturally in this context.

Let $\mathcal{W}$ be a three-dimensional Weyl space i.e. a manifold with with a torsion-free connection D and a conformal metric $[h]$ such that null geodesics of $[h]$ are also geodesics for D. This condition is equivalent to $D_ih_{jk} = \omega_i h_{jk}$ for some one form ω. Here h_{jk} is a representative metric in the conformal class, and the indices $i, j, k, ...$ go from 1 to 3. If we change this representative by $h \longrightarrow \phi^2 h$, then $\omega \longrightarrow \omega + 2\mathrm{d}\ln\phi$. The one-form ω 'measures' the difference between D and the Levi-Civita connection ∇ of h. A tensor object T which transforms as $T \longrightarrow \phi^m T$ when $h_{ij} \longrightarrow \phi^2 h_{ij}$ is said to be conformally invariant of weight m. The formula for a covariant weighted derivative of a one-form of weight m is

$$D_iV_j = \nabla_iV_j + \frac{1}{2}((1-m)\omega_iV_j + \omega_jV_i - h_{ij}\omega_kV^k). \tag{1}$$

The Ricci tensor W_{ij} is related to the Ricci tensor R_{ij} and of ∇ by

$$W_{ij} = R_{ij} + \nabla_i\omega_j - \frac{1}{2}\nabla_j\omega_i + \frac{1}{4}\omega_i\omega_j + h_{ij}\Big(-\frac{1}{4}\omega_k\omega^k + \frac{1}{2}\nabla_k\omega^k\Big).$$

The conformally invariant Einstein–Weyl (EW) condition on $(\mathcal{W}, h, \omega)$ is $W_{(ij)} = Wh_{ij}/3$, or in terms of the Riemannian data:

$$\chi_{ij} := R_{ij} + \frac{1}{2}\nabla_{(i}\omega_{j)} + \frac{1}{4}\omega_i\omega_j - \frac{1}{3}\Big(r + \frac{1}{2}\nabla^k\omega_k + \frac{1}{4}\omega^k\omega_k\Big)h_{ij} = 0, \tag{2}$$

where χ_{ij} is the trace-free part of the Ricci tensor of the Weyl connection, and $r = h^{ij}R_{ij}$. Weyl spaces which satisfy (2) will be called Einstein–Weyl spaces. The EW equations can be regarded as an integrable system. This is because both the twistor theory [5] and the Lax representation [4] exist. One should therefore be able to construct large families of explicit solutions.

In this paper we shall find explicitly all EW spaces with vanishing scalar curvature $W = h^{ij}W_{ij}$. We shall first establish the following result:

Lemma 1 *If the scalar curvature of the Weyl connection vanishes on and EW space $(\mathcal{W}, [h], D)$, then the Faraday two form $F_{ij} := \nabla_{[i}\omega_{j]}$ is null.*

Proof. The Bianchi identities for the curvature of the Weyl connection written in terms of the Levi-Civita connection and ω are [6]

$$\nabla^i F_{ij} + \frac{1}{2}\omega^i F_{ij} + \frac{1}{3}(\nabla_j W + \omega_j W) = 0. \tag{3}$$

Assume $W = 0$ (this is a well defined condition as W is conformally invariant of weight -2). Contracting (3) with ∇^j, and using $\nabla^j\nabla^i F_{ij} = 0$ yields

$$0 = (\nabla^j \omega^i)F_{ij} + \omega^i \nabla^j F_{ij} = F^{ij}F_{ij} - \frac{1}{2}\omega^i\omega^j F_{ij},$$

so F is null.

$\square$

We conclude that there are no non-trivial scalar-flat EW spaces in the Euclidean signature [1]. Non-trivial solutions can be found in the indefinite signature:

Proposition 2 *Let (h, ω) be an Einstein–Weyl structure with vanishing scalar curvature. Then either (h, ω) is flat, or the signature is $(++-)$ and there exist local coordinates $x^i = (y, x, t)$ such that $\omega = y\mathrm{d}t$, and h is given by one of two solutions:*

$$h_1 = \mathrm{d}y^2 + 2\mathrm{d}x\mathrm{d}t + \Big(x[R(t) - \frac{y}{2}] + \frac{1}{48}y^4 + \frac{1}{12}R(t)y^3 + S(t)y\Big)\mathrm{d}t^2, \tag{4}$$

$$h_2 = \mathrm{d}y^2 + 2\mathrm{d}x\mathrm{d}t - \frac{4x}{y}\mathrm{d}y\mathrm{d}t + \Big(\frac{x^2}{y^2} + \frac{xy}{2} + \frac{1}{8}y^4 + R(t)y^2 + S(t)y\Big)\mathrm{d}t^2, \tag{5}$$

where $R(t)$ and $S(t)$ are arbitrary functions with continuous second derivatives.

Proof. Lemma 1 implies that $F_{ij} = \nabla_{[i}\omega_{j]}$ is a closed null two-form. The conformal freedom together with the Darboux theorem imply the existence of coordinates such that $\omega_i = y\nabla_i t$. Therefore $\omega\wedge\mathrm{d}\omega = 0$, and the nullity of F gives $*F\wedge\omega = 0$. We can rewrite Bianchi identity (3) as

$$2(\mathrm{d}*F) + \omega\wedge *F = 0,$$

and deduce $\mathrm{d}*F = 0$. Therefore $\nabla^i F_{ij} = 0$, and $\varepsilon_i{}^{jk}F_{jk} = f(t)\nabla_i t$. Redefining y, t we can set $f(t) = 1$. The most general metric consistent with $\mathrm{d}t = *\mathrm{d}y\wedge\mathrm{d}t$ is

$$h = \mathrm{d}y^2 + 2(\hat{E}\mathrm{d}s + \hat{F}\mathrm{d}y + \hat{G}\mathrm{d}t)\mathrm{d}t,$$

where $\hat{E}, \hat{F}$, and $\hat{G}$ are functions of (s, y, t). Put $\hat{E} = \partial x/\partial s$ and define $G = (\hat{G} - x_t)/2, F = \hat{F} - x_y$, so that

$$h = \mathrm{d}y^2 + 2\mathrm{d}x\mathrm{d}t + 2F\mathrm{d}y\mathrm{d}t + G\mathrm{d}t^2, \qquad \omega = y\mathrm{d}t. \tag{6}$$

The freedom $x \to x + P(y, t)$ implies that $F(x, y, t)$, and $G(x, y, t)$ are defined up to addition of derivatives of $P(y, t)$. Furthermore the conformal scale is only fixed up to arbitrary functions of t, $h \mapsto \tilde{h} = \Omega h$. This leads to to the redefinitions $(x, y, t) \to (\tilde{x}, \tilde{y}, \tilde{t})$, given by

$$\tilde{t} = T(t), \qquad \tilde{y} = \frac{y}{T_t} - 2\frac{T_{tt}}{T_t}, \qquad \tilde{x} = \frac{x}{{T_t}^3} + P(y, t), \qquad \Omega = T_t^2. \tag{7}$$

These transformations will latter be used to simplify F and G. Now impose the EW equations: Equation $\chi_{12} = 0$ implies $F_{xx} = 0$, and we can choose $P(y, t)$ such that $F = xl(y, t)$. Now $\chi_{12} = \chi_{22} =$

0. Equations $\chi_{11}=0, \chi_{23}=0$ are equivalent and imply $a_y = G_{xx}$. Take $G = x^2 l_y/2 + xm(y,t) + n(y,t)$. The vanishing of the scalar curvature

$$W = r + 2\nabla^k \omega_k - \frac{1}{2}\omega^k \omega_k = (3l^2 - 6l_y)/2$$

gives $l^2 = 2l_y$, therefore $l(y,t) = 0$ (case **1**), or $l(y,t) = -2/(y + c(t))$ (case **2**).

- In case **1** $\chi_{31} = 0$ implies $m(y,t) = R(t) - y/2$. Finally $\chi_{33} = 0$ yields $n(y,t) = y^4/48 + R(t)y^3/12 + S(t)y + Z(t))$. Redefining x, we set $Z(t) = 0$, and the metric is given by (4).
- In case **2** the conformal freedom (7) can be used to eliminate $c(t)$. This is achieved by setting $T_{tt} = c(t)$, and redefining $m(y,t)$, and $n(y,t)$. Now $\chi_{31} = 0$ implies $m(y,t) = y/2 + P(t)$, and $\chi_{33} = 0$ gives $n(y,t) = y^4/8 + y^3 P(t)/4 + R(t)y^2 + S(t)y$. In fact we can get rid of $P(t)$: Replace x by $x - y^2 w(t)/2$, and redefine $R(t)$ to obtain (5).

□

The next proposition shows that there doesn't exist a combination of coordinate and conformal transformations which maps (h_1, ω) to (h_2, ω). Cases **1** and **2** are essentially distinct and can be invariantly characterized:

Proposition 3 *Let (h, ω) be a non-flat EW structure with a vanishing scalar curvature, and let F be a corresponding Faraday two form. If $*F$ is parallel with respect to a weighted Weyl connection, then (h, ω) is locally given by (4). Otherwise it is locally given by (5).*

Proof. In both cases **1** and **2** $*F = \varepsilon_i{}^{jk} F_{jk} \mathrm{d}x^i = \mathrm{d}t$. First notice that vanishing of $D(f\mathrm{d}t)$ for some f is invariant condition; in a conformal scale defined by f the one-form $*F$ is parallel. Here we treat $f\mathrm{d}t$ as a weighted object. In case **1** we find that with $m = 3/2$ we have $D(f\mathrm{d}t) = 0$ if $f = f(t)$, and $f_t + fR/2 = 0$. In case **2** $D(f\mathrm{d}t)$ doesn't vanish for any f. For example $(D(f\mathrm{d}t))_{13} = -f/y$.

□

Using the formula for a weighted derivative of a vector of weight m

$$D_i V^j = \nabla_i V^j - \frac{1}{2}\delta_i^j \omega_k V^k - \frac{m+1}{2}\omega_i V^j + \frac{1}{2}\omega^j V_i$$

we deduce that the EW structure (4) (case **1**) admits a covariantly constant vector

$$V = \exp\Big(-\frac{1}{2}\int^t R(t)\mathrm{d}t\Big)\frac{\partial}{\partial x}$$

with weight $-1/2$. Therefore (h_1, ω) belongs to the conformal class of the dKP Einstein–Weyl spaces [4].

It is natural to ask if other special classes of Einstein–Weyl spaces include Lorentzian scalar-flat examples. Along these lines, we have the following observation.

Proposition 4 *Let R be an arbitrary function of one variable. Then the Weyl structure*

$$h = 4\frac{(z + R(v))^2}{(1 + vw)^2}\mathrm{d}v\mathrm{d}w + \mathrm{d}z^2, \qquad \omega = \frac{4}{z + R(v)}\mathrm{d}z \tag{8}$$

is scalar-flat, and $u = 2\log(2(z + R(v))/(1 + vw))$ is a solution of the Lorentzian $SU(\infty)$ Toda equation $4u_{vw} + (e^u)_{zz} = 0$.

Proof. These are straightforward verifications, using the fact that the scalar curvature of the Weyl structure ($h = e^u \mathrm{d}v\mathrm{d}w + \mathrm{d}z^2$, $\omega = 2u_z\,\mathrm{d}z$) is $\frac{1}{2}u_{zz} + \frac{1}{4}u_z^2$: these spaces are Lorentzian analogues of the hyperCR–Toda Einstein–Weyl spaces [2], see also [7].

□

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